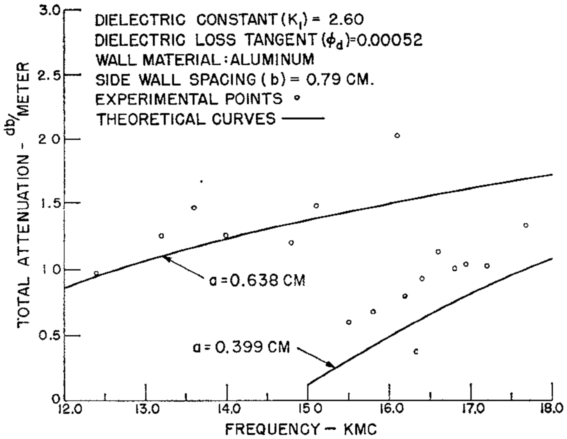

Fig. 11—Comparison of the measured and theoretical total attenuation of the trough line TE_{20} mode as a function of frequency.

and being smaller than the corresponding parallel plate line with comparable attenuation and power handling characteristics. The guide geometry and magnetic field configuration are appropriate for the fabrication of ferrite devices employing transverse magnetization.

The measurements of guide wavelength, rate of field decay and attenuation verify the theoretically predicted properties of this structure.

ACKNOWLEDGMENT

The author would like to thank Dr. E. S. Cassedy for his helpful discussions and criticisms, I. E. Pakulis for his careful carrying out of the measurements, Miss M. D. Velten for the numerical calculations and curve plotting, and J. W. Rodgers and W. Weber for construction of the trough line and associated apparatus.

Coupling of Modes in Uniform, Composite Waveguides*

L. C. BAHIANA† AND L. D. SMULLIN‡

Summary—**The principle of coupling of modes is used to compute the phase constant in a uniform waveguide filled with two different dielectric materials. The natural modes of two hypothetical waveguides filled with the different dielectrics are computed. The propagation of the combined system is computed by considering the coupling between the two sets of modes. Comparison is made between the approximate theory and an exact theory.**

I. INTRODUCTION

THE expression "uniform, composite waveguide" is used in this paper to describe any hollow metallic cylinder of arbitrary cross section filled with two or more homogeneous isotropic materials. Both the structure and the materials are uniform in the direction of propagation. Familiar examples of uniform, composite waveguides are waveguides partly filled with dielectric or magnetic material. The solution of the boundary value problem in such waveguides invariably leads to transcendental equations. Numerical solutions for a few particular cases have been published.[1]

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[1] T. Moreno, "Microwave Transmission Design Data," Dover Publications, Inc., New York, N. Y.; 1958.

A different formulation of the problem is presented here and applied to the case of lossless waveguides containing two media. The fields are expressed in terms of the natural modes of two hypothetical waveguides, found by imposing short-circuit ($\bar{n}\times\overline{E}=0$) or open-circuit ($\bar{n}\times\overline{H}=0$) constraints at the boundary between the two media. Maxwell's equations are then transformed, by the use of conventional techniques, into an infinite set of coupled transmission-line equations. Although this formulation is completely general, its practical usefulness stems from the possibility of obtaining approximate solutions without cumbersome numerical computation.

II. FORMULATION OF THE PROBLEM

A. *Equivalent Current Sheets*

Fig. 1 shows the cross section of a composite waveguide. Surface S_1 is the metallic envelope. Surface S_2 is the boundary between the two media. The solution of Maxwell's equations in medium 1 is unique if either $\bar{n}\times\overline{E}$ or $\bar{n}\times\overline{H}$ is specified over the boundary. The same is true of medium 2. Let $\bar{n}\times\overline{E}_2(S_2)$, where $\overline{E}_2$ is the unknown field in region 2, be specified over S_2. Then we can solve Maxwell's equations in medium 1. In order to do this, surface S_2 is replaced by a metallic wall S (short circuit, with $\bar{n}\times\overline{E}=0$) and a *magnetic* current sheet $\overline{K}_m=\bar{n}\times\overline{E}_2(S_2)$. Mathematically, this is equivalent to transforming a homogeneous differential equation with inhomogeneous boundary conditions into an inhomogeneous differential equation with homogeneous boundary conditions. The solution of Maxwell's equations in medium 1 gives us the tangential magnetic field, $\bar{n}\times\bar{H}_1(S_2)$, at the boundary S_2. Next, we solve Maxwell's equations in medium 2, with $\bar{n}\times\bar{H}_1(S_2)$ specified over S_2. This time, we replace the surface S_2 by a magnetic wall S (open-circuit, with $\bar{n}\times\bar{H}=0$) and an *electric* current sheet $\bar{K}_e=\bar{n}\times\bar{H}_1(S_2)$. By this procedure, we have split the original boundary value problem into one of two conventional waveguides, driven by electric or magnetic surface currents (Fig. 2). The two hypothetical waveguides will be called subwaveguides 1 and 2.

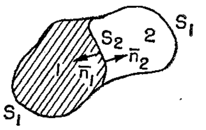

Fig. 1—Composite waveguide of arbitrary cross section.

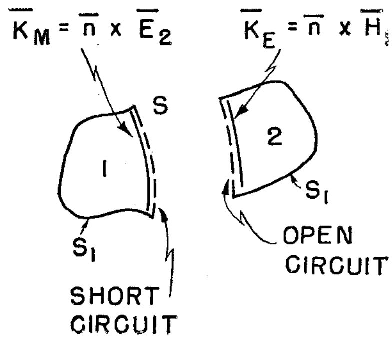

Fig. 2—Equivalent driving currents and boundary constraints.

B. Short-Circuit and Open-Circuit Expansions

The fields in the subwaveguides can be expanded in terms of E-modes and H-modes that satisfy either short-circuit or open-circuit conditions at the common boundary. If the fictitious wall S is short-circuited, the modes will be short-circuit modes; if S is open-circuited, the modes will be open-circuit modes. In either case, the modes are solutions of the scalar Helmholtz equation. These modes form a complete set.

The E-modes are given by the following:

$$\bar{e}_t^e = -\nabla_t\phi,$$
$$\bar{h}_t^e = \bar{k}\times\bar{e}_t^e, \text{ and}$$
$$e_z = \frac{p_e^2}{\Gamma}\phi, \tag{1}$$

where Γ is the propagation constant, $\bar{k}$ is the unit vector in the z-direction, ϕ satisfies the equation $\nabla_t^2\phi+p_e^2\phi=0$, and the boundary conditions are:

$$\phi = 0 \text{ over } S \text{ and } S_1 \qquad \text{for short-circuit modes,}$$
$$\left.\begin{aligned}\phi &= 0 \text{ over } S_1\\ \bar{n}\cdot\nabla_t\phi &= 0 \text{ over } S\end{aligned}\right\} \qquad \text{for open-circuit modes.}$$

The H-modes are given by the following:

$$\bar{h}_t^h = -\nabla_t\psi,$$
$$\bar{e}_t^h = \bar{h}_t^h\times\bar{k}, \text{ and}$$
$$h_z = \frac{p_h^2}{\Gamma}\psi, \tag{2}$$

where Γ is the propagation constant, ψ satisfies the equation $\nabla_t^2\psi+p_h^2\psi=0$, and the boundary conditions are:

$$\bar{n}\cdot\nabla_t\psi = 0 \text{ over } S \text{ and } S_1 \qquad \text{for short-circuit modes,}$$
$$\left.\begin{aligned}\psi &= 0 \text{ over } S\\ \bar{n}\cdot\nabla_t\psi &= 0 \text{ over } S_1\end{aligned}\right\} \qquad \text{for open-circuit modes.}$$

In terms of these modes, the transverse fields in either subwaveguide can be expressed[2] as

$$\bar{E}_t = \sum_{j=1}^{\infty} V_j(z)\bar{e}_{tj}(x, y)$$
$$\bar{H}_t = \sum_{j=1}^{\infty} I_j(z)\bar{h}_{tj}(x, y). \tag{3}$$

This expansion includes both $\bar{E}$ and $\bar{H}$ modes, but the choice of the imaginary wall determines whether they are short-circuit or open-circuit modes.

If we use the orthogonality condition,

$$\int (\bar{e}_{ti}\times\bar{h}_{tj}\cdot\bar{k})da = \delta_{ij}, \tag{4}$$

then the amplitudes V_j and I_j for the jth mode are

$$V_j(z) = \int (\bar{E}_t\cdot\bar{h}_{tj}\times\bar{k})da$$
$$I_j(z) = \int (\bar{H}_t\cdot\bar{k}\times\bar{e}_{tj})da, \tag{5}$$

and the integration is extended over the cross section of the subwaveguide.

C. Solution of Maxwell's Equations in the Subwaveguides

Maxwell's equations for harmonic time-dependence in a loss-free region containing electric and magnetic currents are

$$\nabla\times\bar{E} = -j\omega\mu\bar{H} - \bar{J}_m \text{ and}$$
$$\nabla\times\bar{H} = j\omega\epsilon\bar{E} + \bar{J}_e.$$

After separating the fields into longitudinal and transverse components, the use of (3) yields, for the longitudinal components,

[2] H. A. Haus, "Microwave Circuits," Course 6.621 Class Notes, Mass. Inst. of Tech., Cambridge, Mass.; 1959. (Unpublished.)

$$E_z = \frac{1}{j\omega\epsilon}\sum_{j=1}^{\infty} I_j[\nabla_t\cdot \bar{h}_{tj}\times \bar{k}] - \frac{J_{ez}}{j\omega\epsilon} \text{ and}$$

$$H_z = \frac{1}{j\omega\mu}\sum_{j=1}^{\infty} V_j[\nabla_t\bar{k}\times \bar{e}_{tj}] - \frac{J_{mz}}{j\omega\mu}\cdot \tag{6}$$

By using (3)–(6), the transverse part of Maxwell's equations for the jth mode can be transformed into the following:

$$-\frac{dV_j}{dz} = j\omega\mu_j I_j + \frac{p_{ej}^2}{j\omega\epsilon} I_j + \int (\bar{J}_m\cdot\bar{h}_{tj})da + \frac{\Gamma_{ej}}{j\omega\epsilon}\int (\bar{J}_e\cdot\bar{e}_{zj})da$$

$$-\frac{dI_j}{dz} = j\omega\epsilon V_j + \frac{p_{hj}^2}{j\omega\mu} V_j + \int (\bar{J}_e\cdot\bar{e}_{tj})da + \frac{\Gamma_{hj}}{j\omega\mu_j}\int (\bar{J}_m\cdot\bar{h}_{zj})da. \tag{7}$$

Equations of this form have been derived, for metallic waveguides, by Marcuvitz.[3] They are extended here for open-circuit modes. Detailed proofs are omitted.

D. Coupled Transmission-Line Equations

The coupling equations for a composite waveguide containing two media will now be derived. The subscript j has been retained for the subwaveguide with the short circuit imposed at the wall, and the subscript k has been used for the subwaveguide with the open circuit imposed at the boundary.

For the short-circuited subwaveguide,

$$\bar{J}_m = \bar{K}_m(S) = \bar{n}\times\bar{E}^0(S),\quad \bar{J}_e = 0, \tag{8}$$

where $\bar{E}^0(S)$ is the electric field in the open-circuited subwaveguide at the boundary S.

Similarly, for the open-circuited subwaveguide,

$$\bar{J}_e = \bar{K}_e(S) = \bar{n}\times\bar{H}^s(S),\quad \bar{J}_m = 0. \tag{9}$$

Use of (7)–(9) yields the coupling equations:

$$\left.\begin{aligned} -\frac{dV_j}{dz} &= M_{jj}I_j + \sum_k M_{jk}I_k \\ -\frac{dI_j}{dz} &= N_{jj}V_j + \sum_k N_{jk}V_k \end{aligned}\right\} \text{ and} \tag{10a}$$

$$\left.\begin{aligned} -\frac{dV_k}{dz} &= M_{kk}I_k + \sum_j M_{kj}I_j \\ -\frac{dI_k}{dz} &= N_{kk}V_k + \sum_j N_{kj}V_j \end{aligned}\right\}, \tag{10b}$$

where the sums are over all the modes of the opposite subwaveguide, and the coupling terms M and N are given by

$$\left.\begin{aligned} M_{jj} &\equiv j\omega\mu_j + \frac{p_{ej}^2}{j\omega\epsilon_j} + \frac{1}{j\omega\epsilon_k}\int (\bar{h}_{tj}\cdot\bar{h}_{tj})ds \\ M_{jk} &\equiv \frac{\Gamma_{ek}}{j\omega\epsilon_k}\int (\bar{n}\cdot\bar{e}_{zk}\times\bar{h}_{tj})ds \\ N_{jj} &\equiv j\omega\epsilon_j + \frac{p_{hj}^2}{j\omega\mu_j} \\ N_{jk} &\equiv \frac{\Gamma_{hj}}{j\omega\mu_j}\int (\bar{n}\cdot\bar{e}_{tk}\times\bar{h}_{zj})ds \end{aligned}\right\} \tag{11}$$

and

$$\left.\begin{aligned} M_{kk} &\equiv j\omega\mu_k + \frac{p_{ek}^2}{j\omega\epsilon_k} \\ M_{kj} &\equiv \frac{\Gamma_{ek}}{j\omega\epsilon_k}\int (\bar{n}\cdot\bar{e}_{zk}\times\bar{h}_{tj})ds \\ N_{kk} &\equiv j\omega\epsilon_k + \frac{p_{hk}^2}{j\omega\mu_k} + \frac{1}{j\omega\mu_j}\int (\bar{e}_{tk}\cdot\bar{e}_{tk})ds \\ N_{kj} &\equiv \frac{\Gamma_{hj}}{j\omega\mu_j}\int (\bar{n}\cdot\bar{e}_{tk}\times\bar{h}_{zj})ds \end{aligned}\right\}. \tag{12}$$

The integrals are taken along the boundary, in the transverse plane. The subscripts j and k have been used on μ and ϵ, to avoid inserting a second subscript. They are necessary because μ and ϵ of each subwaveguide appear in the equations for the other. They do not imply that ϵ and μ are different for each mode.

In the following discussion we shall consider only the case of coupling between two modes, one in each subwaveguide.

If we assume propagation of the form $e^{-\Gamma z}$ for the composite system, we obtain

$$\frac{dV_j}{dz} = -\Gamma V_j,\qquad \frac{dI_j}{dz} = -\Gamma I_j,$$

$$\frac{dV_k}{dz} = -\Gamma V_k,\qquad \frac{dI_k}{dz} = -\Gamma I_k. \tag{13}$$

Then the coupling equations (10a) and (10b) can be reduced to

$$[\Gamma^2 - (M_{jj}N_{jj} + M_{jk}N_{kj})]V_j - [M_{jj}N_{jk} + M_{jk}N_{kk}]V_k = 0;$$

$$-[(M_{kk}N_{kj} + M_{kj}N_{jj})]V_j + [\Gamma^2 - (M_{kk}N_{kk} + M_{kj}N_{jk})]V_k = 0. \tag{14}$$

Hence Γ^2 is given by

$$\Gamma^2 = \tfrac{1}{2}[(T_{jj} + T_{kk}) \pm \sqrt{(T_{jj} - T_{kk})^2 + 4T_{jk}T_{kj}}], \tag{15}$$

[3] N. Marcuvitz, "Representation of electric and magnetic fields," *J. Appl. Phys.*, vol. 22, pp. 806–819; June, 1951.

where

$$T_{jj} \equiv M_{jj}N_{jj} + M_{jk}N_{kj},$$
$$T_{kk} \equiv M_{kk}N_{kk} + M_{kj}N_{jk},$$
$$T_{jk} \equiv M_{jj}N_{jk} + M_{jk}N_{kk}, \text{ and}$$
$$T_{kj} \equiv M_{kk}N_{kj} + M_{kj}N_{jj}. \quad (16)$$

III. Applications

In order to obtain good approximations with a few modes (preferably two), care must be taken in the choice of modes. For instance, inspection of (11) and (12) will show that M_{jk} and M_{kj} tend to be small when the dielectric constant of the open-circuit waveguide (ϵ_k) is large. Therefore, if one of the regions has a large dielectric constant, the use of open-circuit modes in this region, and of short-circuit modes in the second region, will insure small coupling between E-modes in the subwaveguides. For dielectric constants larger than 10, the subwaveguides are practically decoupled from each other, and the propagation constants can be found easily. For H-modes, the same advantage is gained if short-circuit modes are used for regions of high μ. In either case, for sufficiently high frequencies, (11) and (12) show that the modes tend to decouple, so that the propagation constants are those of each subwaveguide, taken by itself.

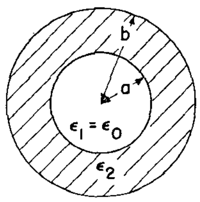

Fig. 3—Circular waveguide half-filled with dielectric material.

A. *Weak Coupling (ϵ_2 large compared with ϵ_1)*

As an example of how these approximations can be found, we shall take the case of a circular waveguide, half-filled with dielectric material (Fig. 3). We are looking for the dispersion characteristics of the circular symmetric TM_{01} (or E_{01}) mode.

Assuming that the two systems are completely decoupled, we solve for the open-circuit modes in the dielectric region. The boundary conditions are:

$$r = b, \qquad E_z = 0;$$
$$r = a, \qquad H_\psi = 0.$$

The solution for the dielectric subwaveguide is found by solving:

$$\Gamma^2 = p^2 - k_2^2 = p^2 - \omega^2\mu_0\epsilon_2;$$

and

$$\frac{J_1(pa)}{N_1(pa)} = \frac{J_0(pb)}{N_0(pb)}.$$

When $b = 2a$, for example, we find that $p = 1.14\pi/b$.[4] It follows that

$$\Gamma^2 = -\beta^2 = -\frac{(2\pi)^2}{\lambda_g^2} = \frac{(1.14\pi)^2}{b^2} - k_2^2$$

$$\beta^2 = \omega\mu_0\epsilon_0\left[\frac{\epsilon_2}{\epsilon_0}\right] - \left[\frac{1.14\pi}{b}\right]^2$$

or

$$\left[\frac{2\pi}{\lambda_g}\right]^2 = \left[\frac{2\pi}{\lambda}\right]^2\epsilon_\gamma - \left[\frac{1.14\pi}{b}\right]^2,$$

where $\epsilon_\gamma \equiv \epsilon_2/\epsilon_0$.

Finally, we obtain the approximate relationship

$$\left[\frac{\lambda}{\lambda_g}\right]^2 = \epsilon_\gamma - 1.3\left[\frac{\lambda}{2b}\right]^2. \quad (17)$$

The exact boundary-value problem has been solved for this case, and the propagation characteristics have been plotted by Marcuvitz.[5] The reader will see that the curves (except for $\epsilon = 2.54$) shown there are described very accurately by (17).

B. *Synchronous Coupling*

For low values of μ and ϵ, the coupling terms cannot be ignored. In order to obtain good approximations in these cases, we take advantage of the fact that when the propagation constants of two modes are approximately equal, the contribution from all other modes may be neglected. If two modes are so chosen that their β-ω characteristics cross each other within the frequency range that is of interest, we can expect good approximations. To illustrate the method, we have calculated the propagation constants of the fundamental mode of a rectangular waveguide half-filled with dielectric of dielectric constant $\epsilon_1 = 2.45\ \epsilon_0$. (See inset, Fig. 4.)

The fundamental mode can be found by coupling the TE_{10} modes of each subwaveguide. The approximation is good if we choose short-circuit modes in the dielectric (subwaveguide 1) and open-circuit modes in the air (subwaveguide 2) because the ω-β curves of these TE_{10} modes cross. Each of these modes is the dominant mode in its respective subwaveguide, and the crossing point is near the cutoff of each. For this simple case, (15) reduces to

$$\Gamma^2 = \tfrac{1}{2}\left[(\Gamma_1^2 + \Gamma_2^2) \pm \sqrt{(\Gamma_1^2 - \Gamma_2^2)^2 + 4K^2}\right],$$

[4] E. Jahnke and F. Emde, "Tables of Functions with Formulae and Curves," Dover Publications, Inc., New York, N. Y., 4th edition, pp. 207–208; 1945.

[5] N. Marcuvitz, "Waveguide Handbook, Radiation Laboratory Series," Vol. 10, McGraw-Hill Publishing Company, Inc., New York, N. Y., p. 395; 1951.

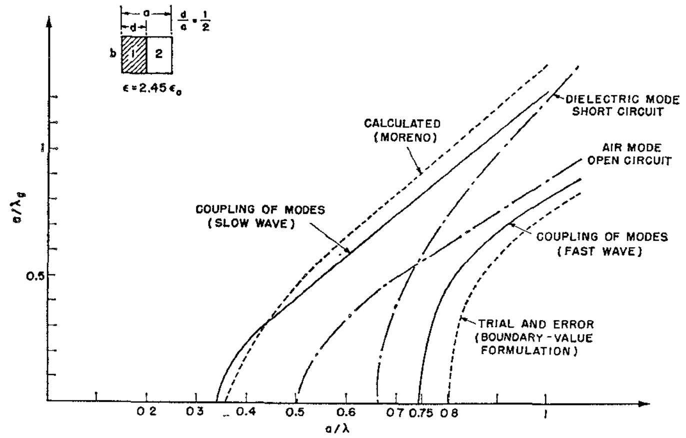

Fig. 4—Propagation in rectangular waveguide half-filled with dielectric material.

where

$$\Gamma_1^2 = p_1^2 - k_1^2 = \left[\frac{\pi}{d}\right]^2 - \left[\frac{\epsilon_1}{\epsilon_0}\right]k_0^2$$

$$k_0^2 = \omega^2\mu_0\epsilon_0,$$

$$\Gamma_2^2 = p_2^2 - k_0^2 = \left[\frac{\pi}{2d}\right]^2 - k_0^2$$

$$K = \frac{\pi}{d\sqrt{d(a-d)}} = \frac{4\pi}{a^2}.$$

A few algebraic transformations lead to

$$\left(\frac{a}{\lambda_g}\right)^2 = -\frac{1}{8}\left\{\left[5 - 13.8\left[\frac{a}{\lambda}\right]^2\right] \pm \sqrt{\left[3 - 5.8\left[\frac{a}{\lambda}\right]^2\right]^2 + 6.5}\right\}.$$

If we solve for the cut-off wavelength, we find that $a/\lambda_c = 0.34$ or $a/\lambda_c = 0.74$. Here we see that we obtain two solutions for $\lambda_g(\lambda)$ as a result of the coupling between the modes in the two subwaveguides. One is a slow wave, and the other a fast wave.

The results of our calculation are plotted in Fig. 4 and compared with the results of an exact computation based on the boundary-value formulation.[6]

ACKNOWLEDGMENT

The authors wish to express their appreciation to Professor H. A. Haus, and to Professor R. L. Kyhl for their many suggestions and stimulating discussions, and to Paul Chorney who read the manuscript and provided helpful criticism.

[6] Moreno, *op. cit.*, p. 192.

CORRECTION

R. C. Johnson, author of "Design of Linear Double Tapers in Rectangular Waveguides," which appeared on pp. 374–378 of the July 1959 issue of these TRANSACTIONS has brought the following corrections to the attention of the Editor.

The first line under (2) should read "where γ_m is the propagation constant in the mth segment."

The expression for b above (7) should be

$$b = b(x) = b_0 + \frac{b_1 - b_0}{L}x.$$

The integral in (14) can be evaluated in closed form; therefore, instead of determining l through the use of (15), it is preferable to use

$$= \frac{L}{2(a_1 - a_0)}\left[\frac{2a_1}{\lambda_{g1}} - \frac{2a_0}{\lambda_{g0}} + \arctan\frac{2a_0}{\lambda_{g0}} - \arctan\frac{2a_1}{\lambda_{g1}}\right],$$

where

$$\lambda_{g0} = \frac{\lambda}{\sqrt{1 - (\lambda/2a_0)^2}}$$

$$\lambda_{g1} = \frac{\lambda}{\sqrt{1 - (\lambda/2a_1)^2}}.$$

The imaginary operator was left out of the exponent term of (19); it should be

$$\Gamma = \frac{i}{8\pi L/\lambda_g}\left[\frac{b_1 - b_0}{b_1}\exp\left(-i4\pi L/\lambda_g\right) - \frac{b_1 - b_0}{b_0}\right]. \quad (19)$$

The close parenthesis symbol was left out of the cosine term in (20); it should be

$$|\Gamma| = \frac{1}{8\pi L/\lambda_g}\left|1 - \frac{b_0}{b_1}\right|\left[1 + \left(\frac{b_1}{b_0}\right)^2 - 2\left(\frac{b_1}{b_0}\right)\cos\left(4\pi L/\lambda_g\right)\right]^{1/2}. \quad (20)$$